Linearity Condition Between a Cavity's Q-Factor and Its Input Resonant Resistance

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***Abstract*— This paper proves that a linear relationship between a cavity's Q-factor and its input resonant resistance holds rigorously true provided the Q variation is caused only by power loss in the cavity.**

I. Introduction

In [1], the authors proved that a cavity Q-factor, Q, is proportional to the normalized input resonant resistance, R, of this cavity, i.e.,

$$\frac{Q}{R} = C \tag{1}$$

where C is a constant provided a moderate perturbation condition is satisfied. This condition implies that the degree of the proportionality between R and Q depends on the degree of the perturbation, suggesting (1) is not a rigorous equation. Moreover, even as an approximate equation, (1) was said to be true only over a small range of Q, as the Q change allowed under a perturbation condition is generally assumed small [2].

However, our recent experimental data, given in Fig. 1, shows that (1) holds true over a large range from about 4000 down to at least 60. This fact suggests that, firstly, (1) may be a rigorous equation because the discrepancies between the measured Q values and those predicted by $Q = CR$ were not significant. (Note that the error in Q measurement alone could be as large as 14% when using a 30 dB directivity coupler in the reflectometer [1]). Secondly, (1) can hold true without imposing a perturbation condition or the assumption of small Q variation.

As both Q and R are important parameters in microwave engineering, and in particular, as (1) allows significantly simplified relative Q measurements by replacing a complex Q measurement with a convenient input resistance measurement (through a reflection coefficient measurement), it is important to know whether (1) is approximate or rigorous and under what condition it holds. In this short paper, using a different approach from that in [1], we prove that (1) is rigorous and holds true as long as the cavity Q variation is caused only by the power loss in the cavity. Consequently, the previously imposed condition of a small Q variation or a moderate perturbation is not necessary [1].

II. Theoretical Proof

By definition

$$Q = \frac{\omega W}{P_L} \tag{2}$$

and

$$R = \frac{V^2}{2P_L} \tag{3}$$

where ω is the angular resonant frequency, P_L is the power loss in the cavity. W is the energy stored in the cavity and V is the equivalent RF voltage at the reference plane where R is defined (see Fig. 2).

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MEASURED R	MEASURED Q	CALCULATED Q BY Q=CR	DIFFERENCE IN Q (%)
6.21	3903	3633	7.00
3.23	1897	1891	0.64
1.97	1153	1153	0.00
1.53	871	893	2.50
1.13	679	663	2.40
0.853	489	498	1.80
0.747	433	437	0.93
0.669	383	391	2.10
0.518	303	303	0.00
0.418	243	245	0.82
0.322	188	189	0.53
0.217	129	127	1.60
0.157	96	92	4.20
0.116	72	68	5.60
0.098	61	58	4.90

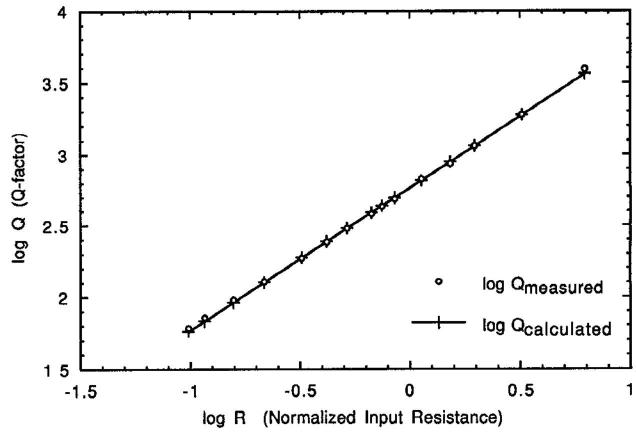

Fig. 1. Measured Q and R values, compared with the calculated Q from $Q = CR$. The table shows the numerical data and the figure plotted from them shows the excellent agreement between the measured Q and the straight line predicted by equation $Q = CR$, demonstrating the true linear relation between Q and R over a large range. ($C = Q/R$ = 1153/1.97 = 585. The measurement techniques used were given in [1]).

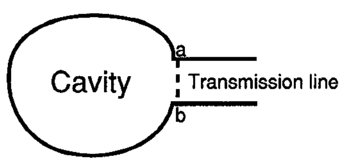

Fig. 2. An arbitrary cavity and its coupling reference plane, ab.

Dividing (2) by (3), we have

$$\frac{Q}{R} = 2\omega \frac{W}{V^2} \tag{4}$$

When resonating, the electric energy, W_e, stored in the cavity equals the stored magnetic energy, W_m. Thus the total energy, W, is

$$W = 2W_e = \frac{\varepsilon}{2} \iiint_V E \cdot E^* \, dv \tag{5}$$

where ε is the permittivity of the medium uniformly filling the cavity. At the reference plane ab (see Fig. 2), the RF voltage, V, can be expressed as

$$V^2 = \left(\int_a^b E \cdot dl \right)^2 \tag{6}$$

Substituting (5) and (6) for W and V_2 in (4), we have

$$\frac{Q}{R} = A\frac{\iiint_V E \cdot E^* \, dv}{\left(\int_a^b E \cdot dl\right)^2} \tag{7}$$

where $A = \omega\varepsilon$, a constant at a given ω and ε. (7) shows that the only requirement to guarantee the proportionality of Q and R, or the validity of (1) is

$$\frac{\iiint_V E \cdot E^* \, dv}{\left(\int_a^b E \cdot dl\right)^2} = B = \text{constant}. \tag{8}$$

In the following, we are to prove that (8) holds no matter how large the Q variation so long as such a change is caused only by a change in the power loss, P_L.

According to [3], P_L and W are both proportional to the square of the field strength; thus P_L at any instant of time is proportional to W, i.e.,

$$P_L = -\frac{dW}{dt} = 2\alpha W \tag{9}$$

where α is an attenuation factor.

The solution of (9) is

$$W = W_0 e^{-2\alpha t} \tag{10}$$

and

$$Q = \frac{\omega W}{P_L} = \frac{\omega}{2\alpha}. \tag{11}$$

Equation (11) shows that the Q is determined by α for a given ω. Because of (10), E at this instant can also be expressed as

$$E = E_0 e^{-\alpha t} \tag{12}$$

where E_0 is the spatial distribution of E. Substituting (12) for E in (8), we obtain

$$B = \frac{\iiint_V E \cdot E^* \, dv}{\left(\int_a^b E \cdot dl\right)^2} = \frac{e^{-2\alpha t}\iiint_V E_0 \cdot E_0^* \, dv}{e^{-2\alpha t}\left(\int_a^b E_0 \cdot dl\right)^2} = B_0 \tag{13}$$

which indicates that the constant B is not a function of α, and, therefore, not a function of Q either for a given frequency. Consequently, any Q variation caused by a power loss variation in the cavity would not alter the constant C in (1). Therefore, (1) holds over an unlimited Q range.

In practice, Q variation can be caused by the variation in the power loss and by the variation in the field spatial distribution, E_0. For the former case, we have proved (1) to hold, However, for the latter case, (1) may or may not be valid depending on whether the constant B is altered as a result of the variation in E_0, for, in essence, validation of (1) only requires invariance of the constant B. Note that B is a ratio of two integrals of E_0; a change in E_0, particularly, a small local change as in all perturbations, does not necessarily change this ratio. Therefore, a linear relation between Q and R holds not only in the situations where the Q variation is caused by the variation in power loss, but also in many situations where the field distribution has somehow been changed but the constant B does not. We have recently applied this concept in the design of a self-heating single-frequency high temperature dielectrometer with excellent results.

References

[1] B. Tian and W. R. Tinga, "Single-frequency relative Q measurement using perturbation theory," *IEEE Trans. Microwave Theory Tech.*, vol. 41, no. 11, pp. 1922–1927, Nov. 1993.

[2] R. A. Waldron, *Theory of Guided Electromagnetic Waves.* London: Van Nostrand Reinhold, 1970. pp. 311–312.

[3] R. E. Collin, *Field Theory of Guided Waves*, 2nd ed. NY: IEEE Press, 1991, p. 388.

Transient Analysis of Lossy Coupled Transmission Lines in a Lossy Medium Using the Waveform Relaxation Method

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Abstract—**The waveform relaxation method has been shown to be both efficient and accurate when applied to coupled transmission lines with conductor losses. In this paper, the method is generalized to include the dielectric loss surrounding the transmission lines. The distributed loss model assumes that the conductance matrix is approximately diagonal and its product with the resistive matrix is a scalar matrix. Computational results using the model is presented and compared with HSPICE solutions.**

I. Introduction

Recently, the method of characteristics has been generalized by Chang [1] for waveform relaxation analysis so that time-domain simulations of lumped-parameter networks interconnected with coupled transmission lines can be carried out more efficiently. It has been shown by the present authors [2] that solution problems related to the presence of dc components can arise, leading to a complete breakdown of the iterative process, and a modified iterative algorithm has been proposed to overcome these problems. In this paper, the dielectric leakage of the medium in which the transmission lines are embedded is taken into consideration.

II. Coupled Lines with Particular Conductor and Dielectric Loss Matrices

Under general conditions, the voltages and currents along a set of lossy coupled transmission lines, each of length l, are described by

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